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The Trace Anomaly and Low Energy Phenomenological Implications of Wormholes

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Abstract

In Coleman's wormhole scenario it is the trace anomaly on S_4 that controls the behavior of fundamental coupling constants, particle masses, mixing angles, etc. We indicate how low energy phenomenology may be deriveable from very general properties of wormhole physics and calculable β–functions, etc.

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Considerable excitement has been generated recently over the idea, due to Coleman, that wormholes in Euclidean spacetime can lead to a relaxation of the cosmological constant [1]. Coleman's scenario further suggests, as applied recently by Grinstein and Wise to a real scalar field [2], how the fundamental particle masses (and presumeably gauge couplings, mixing angles, etc.) might be influenced, possibly even determined, by wormhole physics. In this letter we remark that the essential discriminant which determines whether particle masses are pushed toward zero or pulled toward the Planck (or wormhole) scale is just the gravitational contribution to the trace anomaly (applied to S_4). The work of ref.[2] is a special case which indicates that a single, real (non–conformally coupled) scalar field is pushed toward zero mass; we shall see that the same result holds for spin–2 while the opposite conclusion obtains for spin-1/2 through spin-3/2.

More generally, however, wormholes supply us with a powerful new set of principles which may dictate the full set of low energy parameters. The essential new ingredients are that: (1) at some very high energy scale, M, (the presumed wormhole scale), the fundamental coupling constants, $\lambda_i(M)$, are free parameters since they have a general dependence upon wormhole parameters, α_i, which are themselves free parameters;[1] (2) the effective potential for the parameters is given by the effective action on S_4. S_4 describes the large virtual baby universes in Coleman's scenario which generate the effective potential in which the cosmological constant vanishes and drives the effects described in this paper. The interesting novelty here is that the effective potential on a large S_4 universe of radius r involves the low energy values of the parameters, $\lambda_i(r^{-1})$, of the theory, while the high energy values, $\lambda_i(M)$ are the free parameters; these are connected by the renormalization–group equations. Nontrivial solutions to the extremal conditions will generally emerge, and one has the possibility that the observed low energy parameters will be completely determined. We will give examples of how this works for gauge coupling constants and for the masses of a standard model

[1]The dependence upon the α_i–parameters for fundamental fields would be expected to be essentially universal, dependent only upon the spins and other quantum numbers, and linear in the α_i in the dilute gas approximation.

quarks and leptons.

First we wish to demonstrate the connection between the trace anomaly and the effective potential for masses of elementary fields. With general matter fields, ψ_i, we may consider the effective action as a series expansion in the background curvature:

$$\Gamma = \int d^4x\sqrt{g}\left\{\Lambda - (16\pi G_N)R + : L_0(\psi_i, g) : + \beta R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} + \gamma R_{\mu\nu}R^{\mu\nu} + \delta R^2 + ...\right\} \tag{1}$$

Here we include all terms to dimension–4 and define $: L_0 :$ to have vanishing vacuum matrix element to all orders in a loop expansion (*i.e.*, no contribution to the cosmological constant; this is to be implemented at the presumed absolute minimum of the potential in the case of spontaneous symmetry breaking). Neglecting terms of order $(G_N)^p$ implies that β, γ, and δ are independent of fields, but are generally functions of the mass parameters and coupling constants appearing in L_0 as a consequence of diagrams with internal ψ_i lines that tie onto external gravitons. Note that this expression should be viewed as an effective action at some scale μ; wormhole effects will drive us to consider the far infra–red limit of the theory, so the ψ_i should be viewed as fields that are fundamental on very low energy scales, hence a pointlike field description of particles like the proton and pion should also be valid in this limit.[2]

The dependence in β, γ, and δ upon the masses of elementary particles can be determined in the leading–log approximation by a simple argument, which parallels the derivation of the Callan–Symanzik equation for the scaling behavior of 1PI Green's functions in perturbation theory. We suppose in the tree approximation that β, γ, and δ are just constants (perhaps vanishing, but otherwise arbitrary) with no dependence upon the parameters of L_0. The divergence of the scale–current, S_μ, satisfies the familiar Noether relationship:

$$\partial^\mu S_\mu = T^\mu_\mu. \tag{2}$$

[2] However, only fields that are pointlike on the wormhole scale, M, will have masses that may be viewed as free parameters; we are therefore confused about the conclusion of ref.[3] that the pion will be driven to zero mass since at the wormhole scale it is composed of quarks which are naively driven to large masses.

Let us initially assume that there are no dimensionful parameters contained in L_0. In this case it is well known that, although the trace of the stress-tensor can be chosen to be zero in tree-approximation (scalars must be conformally coupled to gravity as in $\frac{1}{2}\xi\phi^2 R$, $\xi = 1/6$) there will nonetheless be a nonzero T_μ^μ commencing at order $\hbar$. This represents an explicit breakdown of scale-invariance by eq.(2).

Eq.(2) is an operator equation which must hold for all of it's matrix elements. The breaking of scale-invariance in an apparently invariant theory at tree approximation is a consequence of the necessity of introducing a mass parameter, $\mathcal{M}$, to regulate the divergences of the loop expansion of operator matrix elements. Scale-invariance is thus explicitly broken by the regularization, and the trace anomaly is the residual effect of this. Renormalization trades the explicit cut-off dependence, $\mathcal{M}$, for a renormalization-point mass-scale, μ. The renormalized matrix elements of $T_{\mu\nu}$ generally contain implicit μ dependence, and the theory will have broken scale invariance and a nonzero trace.

Now, under scale transformations we have $\psi_i(x) \rightarrow \lambda^{d_i}\psi_i(\lambda x^\mu)$ where d_i is the mass-dimension of ψ_i. Scale transformations are induced on field operators by a generator, $D = \int d^3x\, S_0$, as:

$$\psi_i \rightarrow e^{iD\ln\lambda}\,\psi_i\,e^{-iD\ln\lambda}; \qquad i[D,\psi_i] = (d_i + x^\mu\partial_\mu)\psi_i. \tag{3}$$

where d_i is the canonical mass-dimension of ψ_i. L_0 has no-dimensionful parameters, hence all of its terms are $d = 4$ and under commutation with D we have:

$$i\,[D, L_0(x)] = \frac{d}{d\ln\lambda}L_0(\lambda^{d_i}\psi_i(\lambda x^\mu))\bigg|_{\lambda=1} = (4 + x^\mu\partial_\mu)L_0(x) = \partial_\mu(x^\mu L_0(x)) \tag{4}$$

hence, the action is invariant under the transformation (in the absence of surface terms). The divergence of the scale current is given by the variation of the action with respect to a local transformation:

$$T_\mu^\mu = \partial^\mu S_\mu = -\frac{\delta}{\delta\ln\lambda(x)}\int d^4x L_0(x) \tag{5}$$

and eq.(5) implies that this is the variation of a vanishing surface term, hence the divergence of S^μ is zero in the tree-approximation. However, let us now consider the renormalized operator $L_0(x)$ in eq.(4) which depends upon the renormalization mass-scale μ. μ is held fixed under scale transformations and we thus have:

$$\partial_\mu S^\mu = -\frac{\delta}{\delta \ln \lambda(x)} \int d^4x L_0 = \frac{\partial}{\partial \ln \mu} L_0. \tag{6}$$

This equation simply expresses the fact that the trace anomaly arises due to the presence of μ in the renormalized amplitudes.

Let us see how this works in a well-known example. Consider a pure Yang-Mills theory with coupling constant e and with the vector potential scaled so that all explicit $e-$dependence in $G_{\mu\nu}$ is removed. The Lagrangian reads:

$$L = -\frac{1}{2e^2} \operatorname{Tr} G_{\mu\nu} G^{\mu\nu} \tag{7}$$

Under renormalization (modulo some subtleties which are irrelevant here) the operator L goes to itself with e replaced by the usual running coupling constant defined at a scale μ, *i.e.*, e has become a function of p^2/μ^2. Thus we have:

$$\begin{aligned} T^\mu_\mu &= \frac{\partial}{\partial \ln \mu} L(x) = \frac{1}{e^3} \frac{\partial e}{\partial \ln \mu} \operatorname{Tr} G_{\mu\nu} G^{\mu\nu} \\ &= -\frac{2\beta(e)}{e} L(x) \end{aligned} \tag{8}$$

where we use $\partial e / \partial \ln \mu = \beta(e)$. The last expression is the usual trace anomaly for a pure Yang-Mills theory.

In external gravitational fields we have divergent loops of the matrix elements of $T_{\mu\nu}$ which lead to the gravitational contribution to the trace anomaly. Following [4,5] these may be written in the form:

$$\left\langle T^\mu_\mu \right\rangle = \frac{1}{2880\pi^2} \left\{ \beta' R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa} + \gamma' R_{\mu\nu} R^{\mu\nu} + \delta' R^2 + \ldots \right\} \tag{9}$$

(we shall drop here a term of the form D^2R which plays no role in our present discussion as it vanishes on spaces of constant curvature such as S_4). We give in Table I the values of the coefficients in theories of various spins up to spin-2 for massless fields and in Table II the results for massive fields [4].

We can see how the trace anomaly is related to the dependence of the coefficients, β, γ, and δ, in eq.(1) upon the masses of other fields in the theory. We now suppose that our theory contains one mass parameter in the defining Lagrangian, L_0. Consider the limit in which the single mass parameter m is the largest physical scale in the theory (large compared to the external momenta of the matrix elements of L_0). Since β, γ, and δ are coefficients of dimension-4 terms, they are dimensionless themselves and must have, in the limit, a dependence upon m^2 through the ratio m^2/μ^2. Hence, for these terms we have:

$$\frac{\partial}{\partial \ln \mu} = -\frac{\partial}{\partial \ln m} \tag{10}$$

Using this result and applying eq.(6) and eq.(10) to compute the trace of the stress tensor gives:

$$T^{\mu}_{\mu} \;\; = \;\; -\frac{\partial}{\partial \ln m}\left[L_0(x)\right] + \text{ terms of } O(m^2) \tag{11}$$

In fact, we may take the $m \rightarrow 0$ limit of the rhs of eq.(11) to obtain the trace anomaly in the theory with no mass parameter. Therefore, we may identify the rhs of eq.(9) with eq.(11), and the associated parameters as listed in Table I. Thus, integrating eq.(11) with repect to $\ln m$ gives the effective Lagrangian:

$$\begin{aligned} L_0 \;\; = \;\; & -\frac{1}{2880\pi^2}\Big[(\beta_0 + \beta' \ln(m/\mu))R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} + (\gamma_0 + \gamma' \ln(m/\mu))R_{\mu\nu}R^{\mu\nu} \\ & + (\delta_0 + \delta' \ln(m/\mu))R^2\Big] + \text{invariant terms and higher order in } m^2. \end{aligned} \tag{12}$$

Here β_0, γ_0, and δ_0 are arbitrary constants, independent of m, and we see that the m-dependence is determined by the coefficients in the trace anomaly.

We now project the action of eq.(1) onto the Euclidean 4-sphere, S_4, following ref.[2]. We find:

$$\Gamma(S_4) = \left[-\frac{3}{8G_N^2\Lambda} - \frac{8\pi^2 c}{3} \right] \tag{13}$$

where:

$$c = 24\beta + 36\gamma + 144\delta = c_0 + c_1 \ln(m/\mu) \tag{14}$$

The values of c_1 which obtain for the different spins are given by the trace anomalies and are also quoted in Table I.

The terms of eq.(14) should be viewed as the negative of the potential energy of the system, *i.e.*, the probability distribution in Coleman's α–parameters is given by:

$$Z = \exp\left[\exp\left(\frac{3}{8G_N^2\Lambda} + \frac{8\pi^2}{3}(c_0 + c_1 \ln(m/\mu)) \right) \right] \tag{15}$$

If the mass term appearing in the argument of the logarithm is a general function of the α–parameters then the most probable value of this parameter is determined by letting $m \to \infty$ for positive c_1 and $m \to 0$ for negative c_1. An inspection of Table I indicates that we reproduce the result of ref.[2], namely that a real, minimally–coupled scalar field will tend toward zero mass. For spin–1/2 through spin–3/2 the opposite conclusion holds, while gravity behaves again like a real scalar. These are naive conclusions, however, because the entire interaction structure of the theory is relevant in making the connection between the mass defined at the wormhole scale and the physical value defined at low energies. We shall now examine this aspect in greater detail.

First, it is instructive to consider the limiting case of exactly massless, conformally coupled fields (*e.g.*, massless spin–1/2 or spin–1 particles). Consider the analogues to the terms of eq.(12) which must now involve a seeming infra–red divergence as $m \to 0$. These cannot now be expressed as an integral over a local Lagrangian density, nevertheless their form, $\Delta\Gamma$, can be easily characterized. Since we are interested primarily in large spheres with radius r we consider $\Delta\Gamma$ as a function of $g_{\mu\nu} = r^2 \tilde{g}_{\mu\nu}$

where $\tilde{g}_{\mu\nu}$ corresponds to the sphere of unit radius. Then

$$\Delta\Gamma(g = r^2\tilde{g}, \mu) = F(r\mu, \tilde{g}), \tag{16}$$

that is, conformal invariance is broken again, only by renormalization effects, and therefore $\Delta\Gamma$ depends on r only through the combination $r\mu$, where μ is again the renormalization–point.

For example, at 1–loop order the contribution to $\Delta\Gamma$ is proportional to (see Birrell and Davies [5]):

$$\mathrm{Tr}\left(\ln(r\mu)^2 \tilde{G}_F(x, x')\right) \tag{17}$$

where the nonlocal propagator G_F is evaluated with the background metric $\tilde{g}$. The calculation of the trace of a constant $(1 \cdot \ln(r\mu))$ gives the r–dependent term in the action to be $(8\pi^2 c_1/3)\ln(r\mu)$. Now, the radius r of the sphere plays the role of an infrared cut-off and therefore this result holds as well for massive particles provided their mass is much smaller than $1/r$. Since we are interested in considering the limit of very large radius (*i.e.*, $r^2 \sim 1/G_N\Lambda$, with $\Lambda \to 0$), eventually we will need to consider the opposite limit $m \gg r^{-1}$, so the mass itself regulates the infrared. The effect should be to change $\ln(r\mu)$ into $\ln(\mu/m)$. Hearteningly this is just what was found above.

Moreover, in a theory with massive and truly massless particles, *i.e.*, those for which a symmetry, such as chiral invariance, prevents a mass, the small Λ corrections to the term of order $1/G_N^2\Lambda$ in Γ are dominated by $\ln\Lambda$ contributions due to massless particles. If there are wormhole α–parameters that are left undetermined from the leading term in Γ, then one should consider the effects of massless particles before those of massive ones. This has the possible outcome of fixing dimensionless coupling constants. We now consider briefly how this may occur.

Consider a theory of massless interacting particles with one dimensionless coupling constant e. The effective action on the sphere will have a contribution

$$\Delta\Gamma = \frac{8\pi^2}{3} f(r\mu) + \cdots \tag{18}$$

where the ellipses stand for terms that are independent of r and $f(r\mu)$ is determined through the renormalization–group equation

$$\mu \frac{df}{d\mu} = \gamma \tag{19}$$

At 1-loop $\gamma = c_1$, and there is no new α–parameter dependence in $\Delta\Gamma$. But beyond 1-loop γ is some function of e, $\gamma = \gamma(e)$ and $f(rM) = f(1) + \int_{r^{-1}}^{M} \frac{d\mu}{\mu}\gamma(\bar{e}(\mu))$ where we have introduced the running coupling constant $\bar{e}(\mu)$ satisfying $\mu\frac{d\bar{e}}{d\mu} = \beta(\bar{e})$ and $\bar{e}(M) = e_0$ at the wormhole scale M. For small but nonvanishing cosmological constant, *i.e.*, large fixed radius, we must minimize $f(rM)$ with respect to e_0, holding Λ and M fixed. It is easy to look for an extremum. Writing

$$f(Mr) = f(1) + \int_{\bar{e}(1/r)}^{e_0} \frac{de}{\beta(e)}\gamma(e) \tag{20}$$

we have the extremum condition

$$0 = \frac{\partial}{\partial e_0} f(Mr) \;=\; \frac{\gamma(e_0)}{\beta(e_0)} - \frac{\gamma(\bar{e}(r^{-1}))}{\beta(\bar{e}(r^{-1}))}\frac{\partial \bar{e}(r^{-1})}{\partial e_0} \tag{21}$$

$$= \frac{1}{\beta(e_0)}\left[\gamma(e_0) - \gamma(\bar{e}(r^{-1}))\right] \tag{22}$$

If $\beta > 0$, nontrivial solutions always exist for γ of the form given in Fig. 1. Only for γ shown in Fig. 1a is the extremum a local minimum. In either case $\bar{e}(1/r) \to 0$ as $r \to \infty$ and therefore e_0 is a solution to $\gamma(e_0) = \gamma(0) = c_1$. Unfortunately, it is generally inconsistent to solve $\gamma(e_0) = \gamma(0)$ perturbatively. Moreover, graviton loops will presumably add a term to the left hand side of eq.(19) proportional to f. To be fair, this is only a toy example[3], but it raises the possibility that Coleman's "big fix" may eventually yield the value of the fine structure constant!

How do wormhole effects influence the masses of standard model quarks and leptons? First we observe in Table II that, naively, fermions will be pulled to large

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[3]In a realistic theory, such as QED with an electron mass m, one expects to replace $\bar{e}(r^{-1})$ by $\bar{e}(m)$ in eq. (22), but we have not presently analyzed this case.

masses. This raises a puzzle in the context of the standard model in which fermion masses are given by a coupling constant (*e.g.*the Higgs–Yukawa coupling) times the VEV which breaks the electroweak symmetry. In particular, there are well known bounds, such as the renormalization–group bounds relevant here [6, 7], which provide upper limits on fermion masses in the standard model, and we must inquire as to how the wormhole scenario either respects or modifies these limits.

We consider a single standard model "top" quark and assume presently that the VEV, v, of the single Higgs is constant and that the strong coupling constant has the usual running; then the physical t–quark mass is $m_t = g_t(m_t) \cdot v$.[4]. We neglect the electroweak couplings g_1 and g_2 presently. The argument of the log in the potential of eq.(13) is then $g_t v/\mu$. Thus, we may write the renormalization–group equation for c in the form:

$$\frac{dc}{d\ln\mu} = \frac{\partial c}{\partial \ln\mu} + \beta_t(g_t, g_3)\frac{\partial c}{\partial g_t} + \beta_3(g_3, g_t)\frac{\partial c}{\partial g_3} = \frac{c_1}{32\pi^2} \tag{23}$$

where:

$$\beta_t(g_t, g_3) = \frac{g_t}{16\pi^2}\left(\frac{9}{2}g_t^2 - 8g_3^2\right); \qquad \beta_3(g_3, g_t) = -b_0\frac{g_3^2}{16\pi^2} \tag{24}$$

to the order of interest. Using these beta–functions one can in principle integrate eq.(23) and obtain an explicit expression of the form:

$$c = \frac{c_1}{32\pi^2}F(g_t(m), g_3(m); g_t(M), g_3(M)) \tag{25}$$

For example, if we neglect the running of g_3 we obtain:

$$F = \frac{\pi^2}{g_3^2}\ln\left[\frac{g_t^2(M)(9g_t^2(m) - 16g_3^2)}{g_t^2(m)(9g_t^2(M) - 16g_3^2)}\right] \tag{26}$$

The condition that $c_1 > 0$ implies that we must choose $g_t(M)$ to maximize F. Here we identify M with the wormhole scale and m with a low energy scale of order m_t.

[4]At present we give no argument to indicate how wormholes might fix the electroweak scale, so the assumption of fixed v is a strong one, but perhaps not unreasonable given that the mechanism fixing v may be independent of that for g_t

However, we readily see from eq.(23) that F must be given equivalently by:

$$F(g_t(m), g_3(m); g_t(M), g_3(M)) = \ln(m/M) \tag{27}$$

The condition that $c_1 > 0$ implies that $\ln(m/M)$ must be made as large as possible by suitable choice of $g_t(M)$. Stated differently, we are holding v fixed, so we must find that initial value (at the wormhole scale) of g_t such that the running time down to the physical mass, $g_t(m_t)$ is minimized (thus we have a kind of "Fermat's Principle" for the Higgs–Yukawa couplings). It is seen from the renormalization–group evolution of Higgs–Yukawa couplings [7] that, for arbitrarily large initial values of $g_t(M)$, we are driven to universal low–energy values. The running time is minimized when the initial $g_t(M)$ is arbitrarily large and the running terminates on the largest value of m_t. Therefore, this predicts that the top quark is driven toward the renormalization–group fixed point value of $g_t(m)$, which is also the upper limit of the running [6, 7]. This is essentially the "triviality bound" of the electroweak theory when we demand that the theory be point–like up to the wormhole scale. Thus, while the fermion is pulled to a large value, it is only pulled to the largest value consistent with the constraint that the underlying theory have no phase transition between its low energy scale $\sim v$ and the wormhole scale, $\sim M_P$. For a fourth generation this corresponds to a quark mass scale of order 220 Gev and the charged lepton of ~ 60 to ~ 110 GeV [7]. However, things can change dramatically when we go beyond the leading order.

To higher order in the loop expansion the right hand side of eq. (23) should be replaced by γ, a function of the coupling constants g_t and g_3. Neglecting the running of g_3 we then obtain

$$c(m_t) = c(M) + \int_{\bar{g}_t(M)}^{\bar{g}_t(m_t)} dg_t \frac{\gamma}{\beta_t} \tag{28}$$

It is straightforward to find a general extremum condition with respect to $g_0 \equiv \bar{g}_t(M)$:

$$0 = \frac{\partial c}{\partial g_0} = \frac{1}{\beta_t(g_0)} \left[\frac{\gamma(\bar{g}_t(m_t))}{1 - \beta_t(\bar{g}_t(m_t))/\bar{g}_t(m_t)} - \gamma(g_0) \right] \tag{29}$$

This equation is to be solved for g_0 holding $v = m_t/\bar{g}_t(m_t)$ fixed. For example, for the functions $\gamma(g)$ and $\gamma(g)/(1-\beta_t(g)/g)$ depicted in Fig. 2, one may always find a solution. In the limit of the 1-loop approximation we recover the preceding result, but this general behavior may yield a range of other interesting solutions. To carry out a complete analysis one requires the two-loop radiative corrections to the trace anomaly.

In conclusion, we have extended the program of ref.[1] and [2] further in the direction of making contact with the low energy phenomenology. We have made no progress here in understanding the origin of the small electroweak scale relative to the wormhole scale. In fact, though scalars by themselves become light as in [2], the standard model Higgs will have larger opposing effects coming from fermions *e.g.*, the $\ln m_f$ terms contain $\ln v \sim \ln m_H$ through the Higgs-Yukawa couplings, with coefficients of opposite sign to the pure scalar case. The effect of gauge bosons is dominant and these also act to change the sign of c_1; the remedy may be to choose very large values of ξ at the wormhole scale or to invoke new interactions so that the Higgs is composite, as in technicolor. We should further mention that considerations as in [8] would be, if applicable, disastrous for the present scenario. We will return to these issues in a more extensive analysis elsewhere.

We thank W. Bardeen, M. Duff, and L. Susskind for useful discussions and comments.

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Table I. Coefficients in trace–anomaly from Christensen and Duff for massless fields of spin–0 though spin–2. Our notation differs from ref.[4] and the translation is $\beta' = a$; $\gamma' = b - 2a$; $\delta' = d - b/3 + a/3$, obtained using the identity $C_{\mu\nu\lambda\kappa}C^{\mu\nu\lambda\kappa} = R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} - 2(R_{\mu\nu}R^{\mu\nu}) + R^2/3$. Here $c_1 = 24\beta' + 36\gamma' + 144\delta'$ as defined in eq.(15). The spin–$\frac{1}{2}$ case is Weyl (multiply by 2 for Dirac). The spin–$\frac{3}{2}$ and spin–2 cases are valid only on mass–shell.

spin	$2880\pi^2\beta'$	$2880\pi^2\gamma'$	$2880\pi^2\delta'$	$32\pi^2 c_1$
0	-1	1	$-90\left(\xi - \frac{1}{6}\right)^2$	$\frac{2}{15} - (12\xi - 2)^2$
1/2	$-\frac{7}{4}$	-2	$\frac{5}{4}$	$\frac{11}{15}$
1	13	-88	25	$\frac{124}{15}$
3/2	$\frac{233}{4}$	$-\frac{233}{2}$	$\frac{649}{24}$	$\frac{61}{5}$
2	-212	424	$-\frac{717}{4} - \frac{212}{3}$	$-\frac{1434}{5}$

Table II. Coefficients in trace–anomaly from Christensen and Duff for massive fields of spin–0 though spin–2, as in Table I. For a massive Dirac field multiply the spin–$\frac{1}{2}$ result by 2. The spin–$\frac{3}{2}$ and spin–2 cases are strictly valid only on mass–shell.

spin	$2880\pi^2\beta'$	$2880\pi^2\gamma'$	$2880\pi^2\delta'$	$32\pi^2 c_1$
0	-1	1	$-90\left(\xi - \frac{1}{6}\right)^2$	$\frac{2}{15} - (12\xi - 2)^2$
1/2	$-\frac{7}{4}$	-2	$\frac{5}{4}$	$\frac{11}{15}$
1	12	-87	$\frac{45}{2}$	$\frac{22}{5}$
3/2	$\frac{226}{4}$	$-\frac{237}{2}$	$\frac{679}{24}$	$\frac{194}{15}$
2	$-211 - \frac{7}{4}$	421	$-268 - \frac{212}{3}$	$-\frac{1411}{5}$

Figure Captions

Figure 1: Two possible forms of the function, γ, in eq.(19) for which solutions to the extremum condition in eq.(22) exist. Only for γ shown in Fig. (a) is the extremum a local minimum.

Figure 2: Possible forms of the "beta-function," γ in eq.(28) and the corresponding form of $\gamma/(1-\beta_t/g)$, for which a solution to the extremal condition in eq.(29) exists. The solution yields the mass of the "top"-quark.

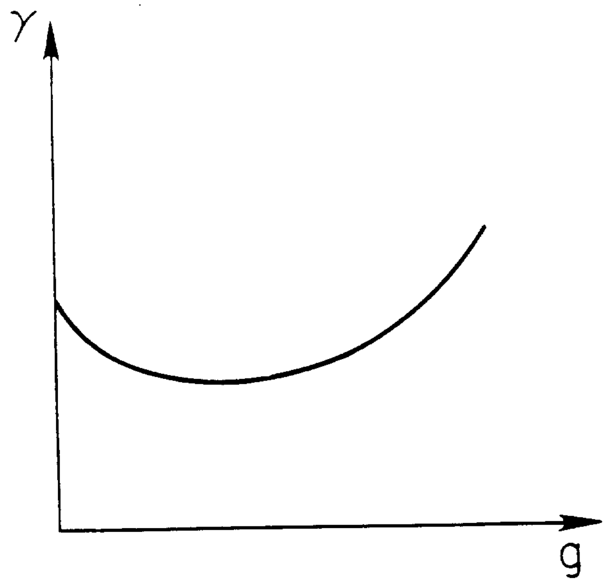

Fig. 1A

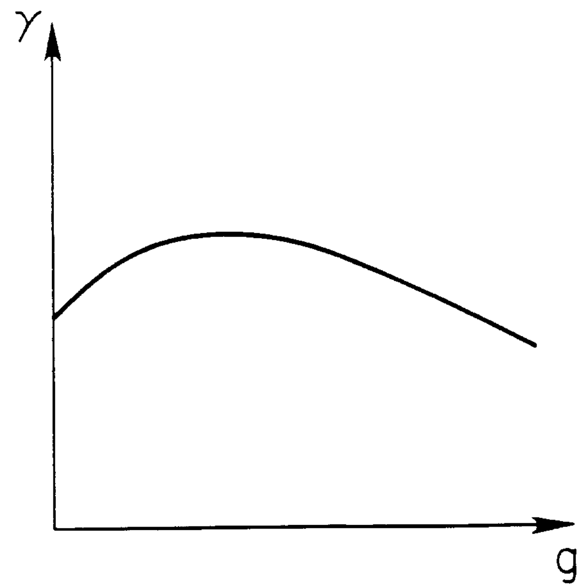

Fig. 1B

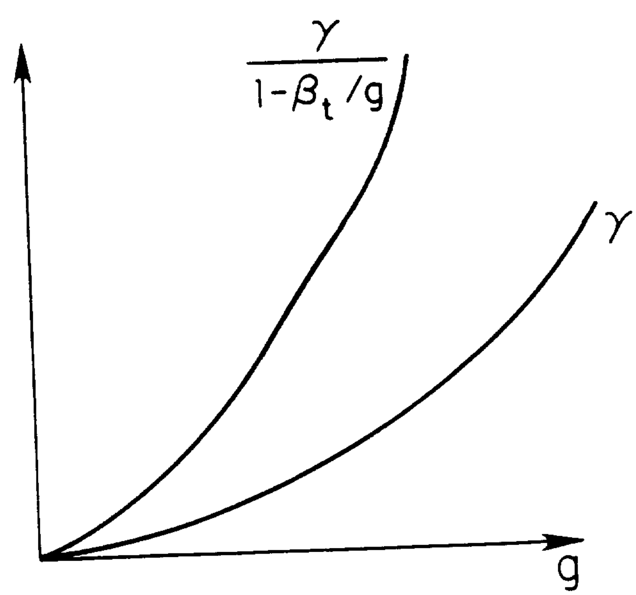

Fig. 2